Variable Structure Control ~ Disturbance Rejection

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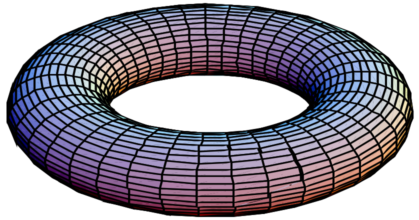

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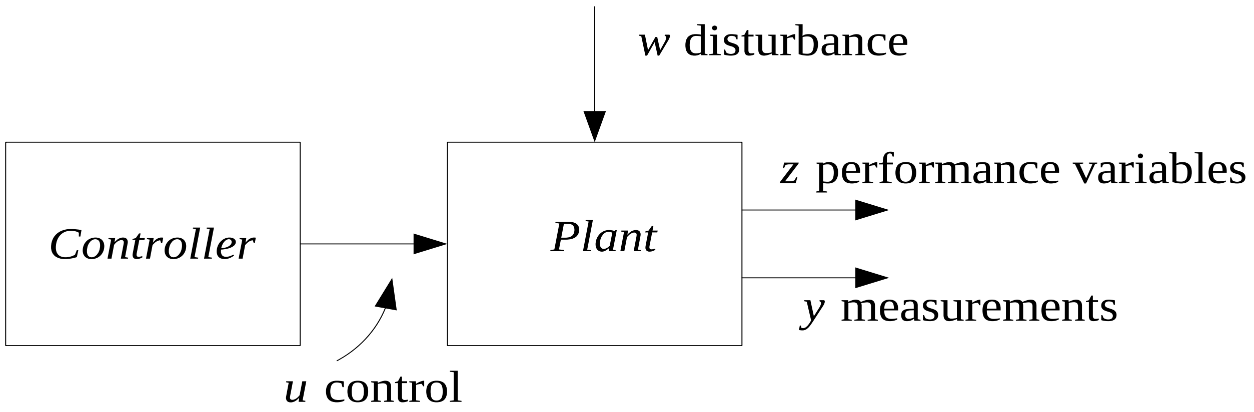
Outline

- Linear Tracking & Disturbance Rejection
- Variable Structure Servomechanism
- Nonlinear Tracking & Disturbance Rejection

Disturbance Rejection Setup



state equations	$\dot{x} = Ax + Ew + Bu$
disturbance	$\dot{w} = Zw$
performance	$z = Cx + Fw + Du$
measurements	$y = \bar{C}x + \bar{F}w + \bar{D}u$

Eigenvalues of Z ordinarily on Im axis

In effect, the error

$$x \in R^n, w \in R^q, u \in R^m, z \in R^m, y \in R^r$$

Objectives

Regulator Problem: Find a controller to achieve the following

1) Regulation: $z(t) \to 0$ as $t \to \infty$

2) (Internal) Stability: Achieve specified transient response

Robust Regulator Problem: Find a solution to the Regulator Problem that satisfies

3) Robustness: 1) and 2) should be maintained under specified small perturbations of plant and/or control parameters

Solution: Part 1- Regulation

Consider the possibility of a control $\bar{u}(t)$ that produces a trajectory $\bar{x}(t)$ for some unspecified initial state $\bar{x}_0$ and any initial disturbance vector w_0, so that the corresponding $\bar{z}(t) \equiv 0$. Then, $\bar{x}, \bar{u}, w$ must satisfy

$$\dot{\bar{x}} = A\bar{x} + Ew + B\bar{u}$$
$$\dot{w} = Zw$$
$$0 = C\bar{x} + Fw + D\bar{u}$$

Assume a solution of the form: $\bar{x} = Xw, \bar{u} = Uw \Rightarrow$

$$\left.\begin{aligned} XZw &= AXw + Ew + BUw \\ 0 &= CXw + Fw + DUw \end{aligned}\right\} \quad \forall w$$

Thus, the hypothesized control $\bar{u}(t)$ exists if there are X, U that satisfy

$$\boxed{\begin{aligned} -XZ + AX + BU &= -E \\ CX + DU &= -F \end{aligned}}$$

Solutions typically are not unique

Solution: Part 2- Stability

Define $\delta u, \delta x$

$$\left.\begin{aligned} u &= \bar{u} + \delta u = Uw + \delta u \\ x &= \bar{x} + \delta x = Xw + \delta x \end{aligned}\right. \Rightarrow \delta \dot{x} = A\delta x + B\delta u$$

Now, if (A, B) is controllable, it easy to choose $\delta u = K\delta x$

so that the closed loop $\delta \dot{x} = (A + BK)\delta x$ has desired transient characteristics.

With K chosen, the control can be written as a function of the system states x, w

$$u = \bar{u} + \delta u = Uw + \delta u = Uw + K\delta x \Rightarrow$$

$$\boxed{u = Uw + K(x - Xw) = \begin{bmatrix} K & U - KX \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = K_{TOT} \begin{bmatrix} x \\ w \end{bmatrix}}$$

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Solution: Part 3- Observation

The control will be implemented using estimates of the composite state (x, w). Consider the composite system

$$\frac{d}{dt}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$y = \bar{C}x + \bar{F}w$$

If the composite system is observable, we can choose a matrix L, so that the following observer has the desired dynamics:

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} = \begin{bmatrix} A & E \\ 0 & Z \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + L\left(\bar{C}\hat{x} + \bar{F}\hat{w} - y\right)$$

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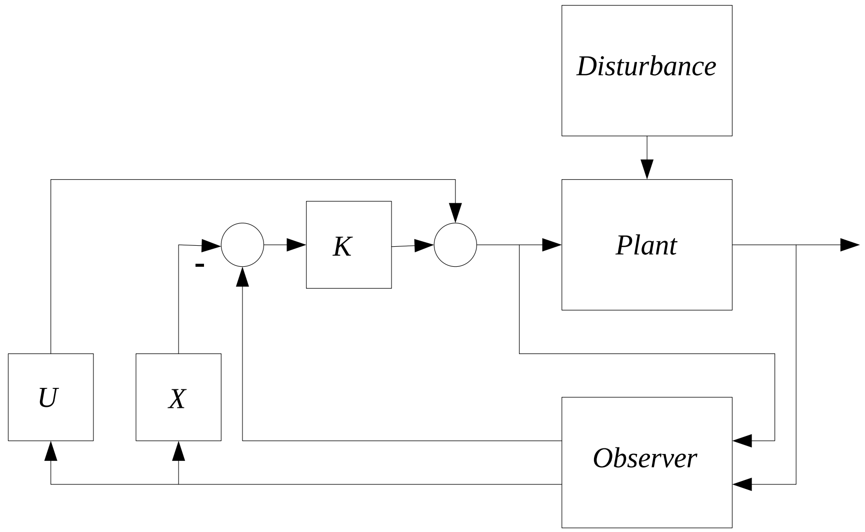
Properties of the Loop



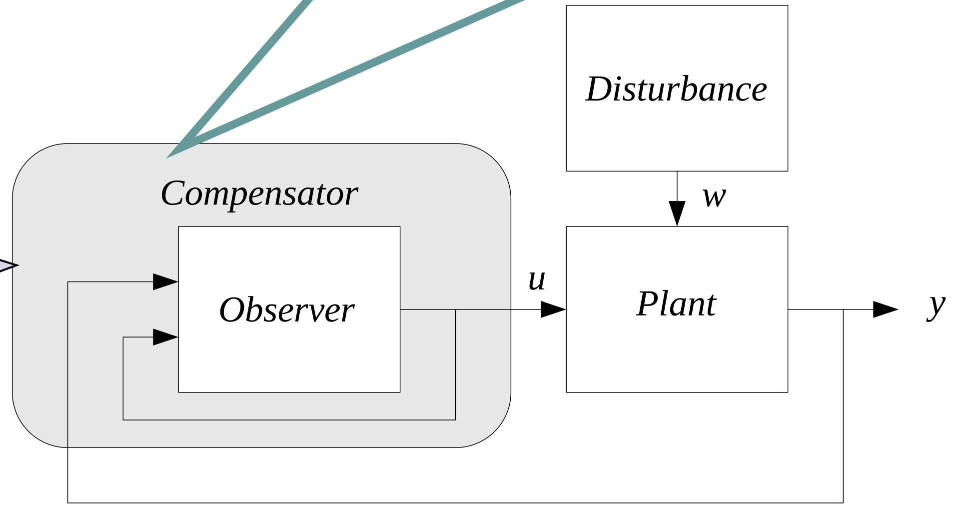
$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \left(\begin{bmatrix} A & E \\ 0 & Z \end{bmatrix} + L \begin{bmatrix} \bar{C} & \bar{F} \end{bmatrix} \right) \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u - Ly$$

$$u = \begin{bmatrix} K & U - KX \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}$$

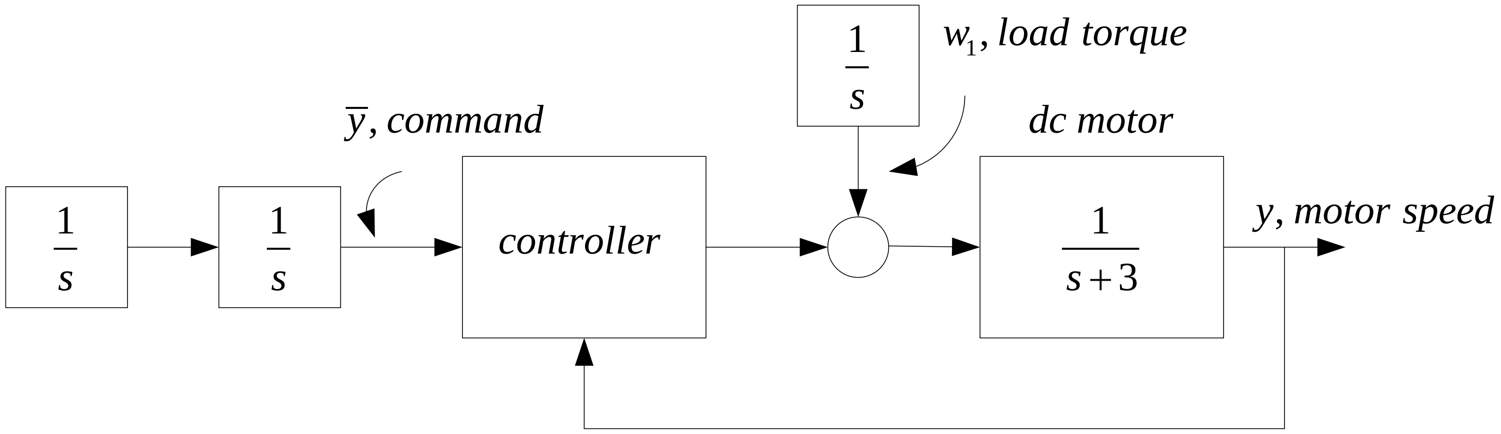
$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \left(\begin{bmatrix} A & E \\ 0 & Z \end{bmatrix} + L \begin{bmatrix} \bar{C} & \bar{F} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K & U - KX \end{bmatrix} \right) \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} - Ly$$

$$u = \begin{bmatrix} K & U - KX \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}$$

Compensator contains copy of Z ~ internal model of disturbance

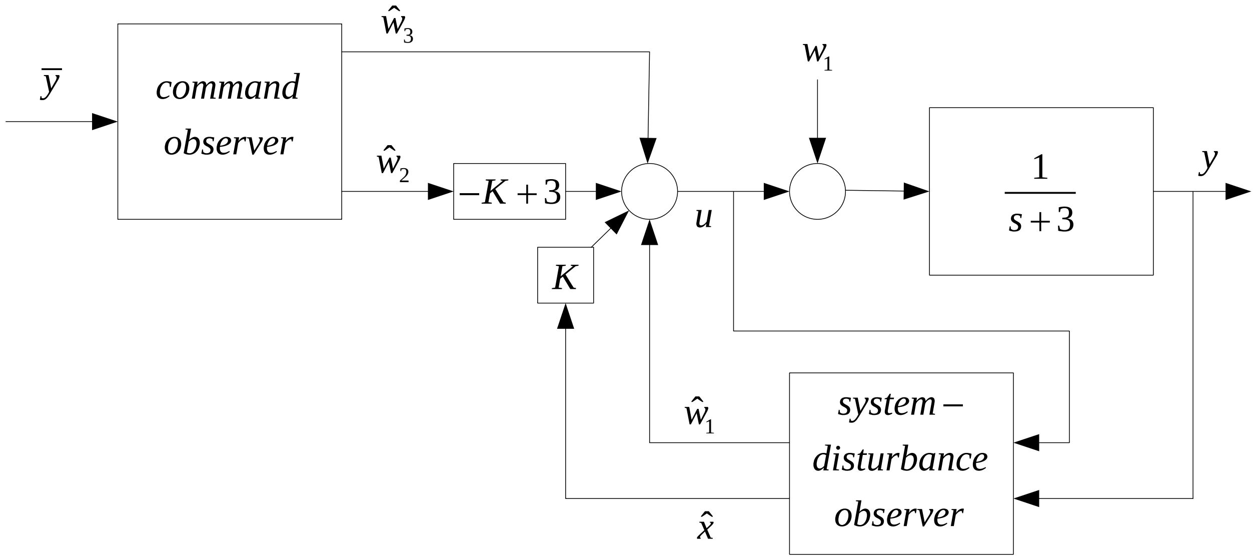


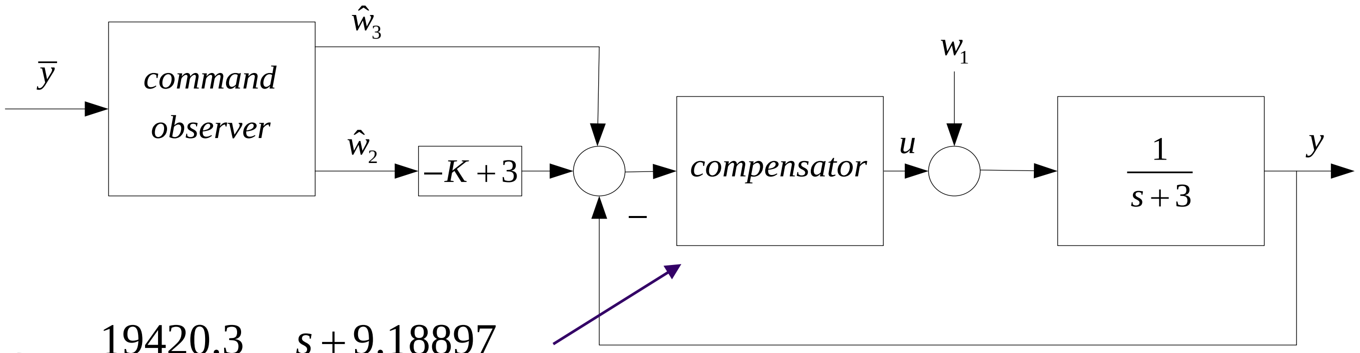
Example



$\frac{1}{s}$
$\frac{1}{s}$
$\bar{y}$, command
controller
$\frac{1}{s}$
w_1, load torque
dc motor
$\frac{1}{s+3}$
y, motor speed

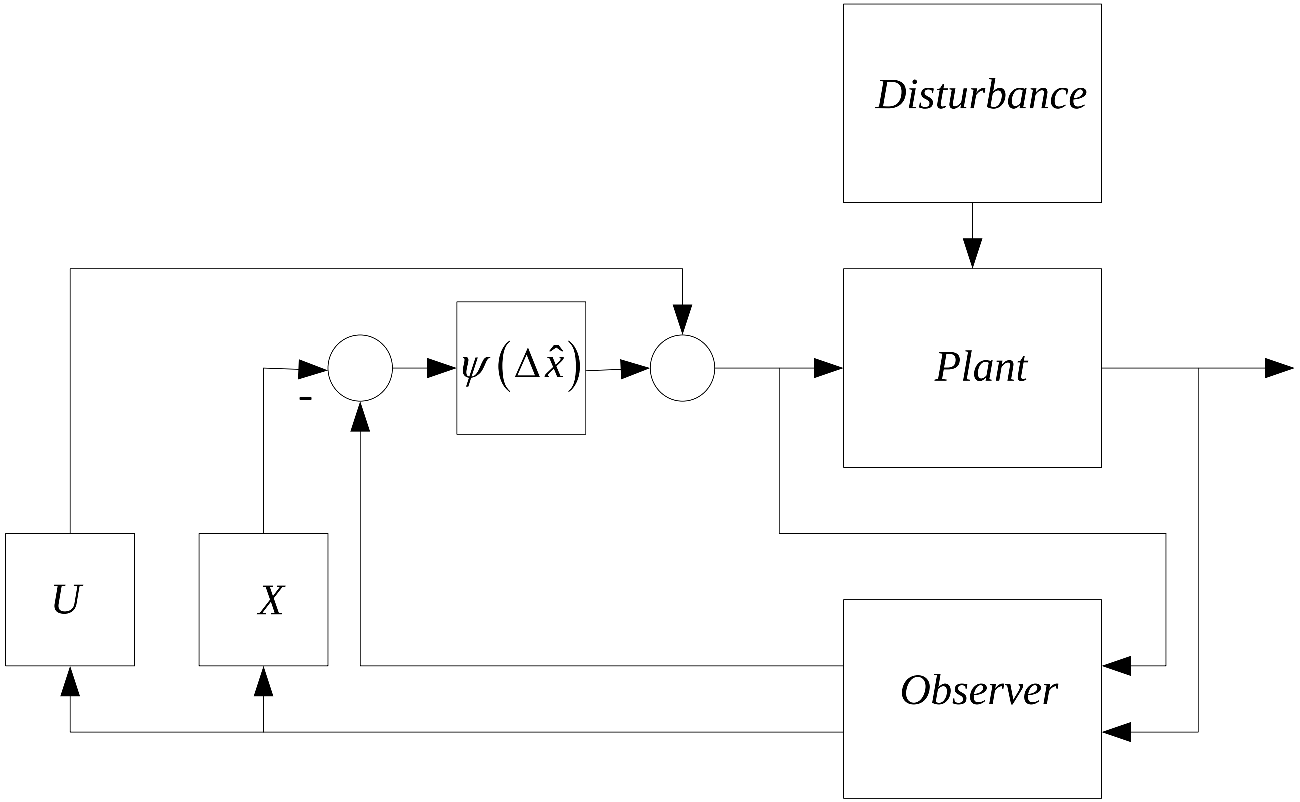
Example





$$G_c = \frac{19420.3}{3+\sqrt{209}} \frac{s+9.18897}{s\left(s+49.8393\right)}$$

VS Servo



We make only one change

$$u = K\Delta\hat{x} \Rightarrow u = \psi(\Delta\hat{x}), \quad \psi_i(\Delta\hat{x}) = \begin{cases} \Delta u_i^+ & s_i(\Delta\hat{x}) > 0 \\ \Delta u_i^+ & s_i(\Delta\hat{x}) < 0 \end{cases}, \quad s(\Delta\hat{x}) = G\Delta\hat{x}$$

Closed Loop Dynamics

Combine the estimator equation:

$$\frac{d}{dt}\hat{x} = A\hat{x} + Ew + Bu + L_1\bar{C}(\hat{x} - x) + L_1\bar{F}(\hat{w} - w)$$

with the estimator equation

$$\frac{d}{dt}\hat{w} = Z\hat{w} + L_2\bar{C}(\hat{x} - x) + L_2\bar{F}(\hat{w} - w)$$

$$\frac{d}{dt}(\hat{x} - X\hat{w}) = A\hat{x} + Ew - XZ\hat{w} + (L_1 - XL_2)\begin{bmatrix}\bar{C} & \bar{F}\end{bmatrix}\begin{bmatrix}\hat{x} - x \\ \hat{w} - w\end{bmatrix} + BU\hat{w} + B\delta u$$

apply $XZ\hat{w} = AX\hat{w} + E\hat{w} + BU\hat{w}$

$$\frac{d}{dt}(\hat{x} - X\hat{w}) = A(\hat{x} - X\hat{w}) + E(w - \hat{w}) + (L_1 - XL_2)\begin{bmatrix}\bar{C} & \bar{F}\end{bmatrix}\begin{bmatrix}\hat{x} - x \\ \hat{w} - w\end{bmatrix} + B\delta u$$

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Closed Loop Dynamics, 2

system

$$\frac{d}{dt}\begin{bmatrix} \hat{x}-\hat{\bar{x}} \\ x-\hat{x} \\ w-\hat{w} \end{bmatrix} = \begin{bmatrix} A & \left(L_1 - XL_2\right)\bar{C} & \left(L_1 - XL_2\right)\bar{F} \\ 0 & A - L_1\bar{C} & E - L_1\bar{F} \\ 0 & -L_2\bar{C} & Z - L_2\bar{F} \end{bmatrix}\begin{bmatrix} \hat{x}-\hat{\bar{x}} \\ x-\hat{x} \\ w-\hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}\delta u$$

Estimator
error

Sliding Behavior

Let $\delta\hat{x} = \hat{x} - \hat{\bar{x}}$ and define a new coordinates

$$\delta\hat{x} \mapsto (\delta\chi_1, \delta\chi_2), \delta\chi_1 \in R^{n-m}, \delta\chi_2 \in R^m$$

$$\begin{bmatrix} M \\ K \end{bmatrix} = \begin{bmatrix} N & B \end{bmatrix}^{-1}$$

$$\delta\hat{x} = N\delta\chi_1 + B\delta\chi_2 \Leftrightarrow \begin{bmatrix} \delta\chi_1 \\ \delta\chi_2 \end{bmatrix} = \begin{bmatrix} M \\ (KB)^{-1} K \end{bmatrix} \delta\hat{x}$$

Note: $MB = 0, KN = 0,$ sliding $\Leftrightarrow \delta\chi_2 \equiv 0 \Leftrightarrow K\delta\hat{x} = 0$

$$\frac{d}{dt}\begin{bmatrix} \delta\chi_1 \\ x - \hat{x} \\ w - \hat{w} \end{bmatrix} = \begin{bmatrix} MAN & M(L_1 - XL_2)\bar{C} & M(L_1 - XL_2)\bar{F} \\ 0 & A - L_1\bar{C} & E - L_1\bar{F} \\ 0 & -L_2\bar{C} & Z - L_2\bar{F} \end{bmatrix} \begin{bmatrix} \delta\chi_1 \\ x - \hat{x} \\ w - \hat{w} \end{bmatrix}$$

Reaching Behavior

Assume the switching control ψ is designed to stabilizes the switching manifold $K\delta x = 0$ for the perturbation system

$$\delta\dot{x} = A\delta x + B\psi(s),\ s = K\delta x$$

Two important results follow

- trajectories are steered in finite time to a domain $\mathfrak{D}$ that contains the subspace $K\delta\hat{x} = 0$
- $\mathfrak{D}$ shrinks exponentially to the subspace $K\delta\hat{x} = 0$

Reaching, 2

Theorem : For a fixed $\Delta > 0$, there exists a finite time $T(\Delta)$ such that $\delta\hat{x}$ is confined to the domain

$$\left|k_i^T \delta\hat{x}\right| \le \Delta,\ i = 1,\ldots,m,\ \forall t \ge T(\Delta).$$

Moreover, $\Delta \to 0,\ T(\Delta) \to \infty.$

- this means that sliding does not occur on $s = 0$ but this manifold is reached asymptotically as $\hat{x}(t) \to x(t), \hat{w}(t) \to w(t)$
- let $\delta\hat{x}^*(t)$ denote an ideal sliding solution. $\delta\hat{x}(t)$ can be viewed as non-ideal sliding in that it can be shown that there exists a constant c such that for all $t \ge T(\Delta)$

  $$\left\|\delta\hat{x}^*(t) - \delta\hat{x}(t)\right\| \le c\Delta$$
- The performance variables can be expressed as

  $$z = C(x - \hat{x}) + F(w - \hat{w}) + D\delta u$$

  as $t \to \infty$, we have $(x - \hat{x}) \to 0, (w - \hat{w}) \to 0, \delta u \to \delta u_{eq} \to 0 \Rightarrow z \to 0$